

BUHEP-01-31
CPT-2001/PE.4294

A Study of the 't Hooft Model with the Overlap Dirac Operator

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Abstract

We present the results of an exploratory numerical study of two dimensional QCD with overlap fermions. We have performed extensive simulations for $U(N_c)$ and $SU(N_c)$ color groups with $N_c = 2, 3, 4$ and coupling constants chosen to satisfy the 't Hooft condition $g^2 N_c = \text{const} = 4/3$. We have computed the meson spectrum and decay constants, the topological susceptibility and the chiral condensate. For $U(N_c)$ gauge groups, our results indicate that the Witten-Veneziano relation is satisfied within our statistical errors and that the chiral condensate for $N_f = 1$ is compatible with a non-zero value. Our results exhibit universality in N_c and confirm once more the excellent chiral properties of the overlap-Dirac operator.

1 Introduction

Several years ago, in a pioneering investigation, 't Hooft studied $U(N_c)$ gauge theories in the limit $N_c \rightarrow \infty$ with $C_t = g^2 N_c$ kept constant [1]. He showed that only planar diagrams with quarks at the edges dominate and therefore some non-perturbative QCD physical observables can be computed in this limit. He proposed two dimensional models [2] with important features of QCD, but simple enough to sum explicitly all planar diagrams in the meson spectrum computation.

Recently, the overlap formulation [3, 4] has made it possible to introduce a lattice Dirac operator D which preserves a lattice form of chiral symmetry at finite cut-off [5]. As a consequence the $U_A(1)$ chiral anomaly is recovered à la Fujikawa [6] and the Dirac Operator has exact chiral zero modes for topologically non-trivial background configurations [7]. A precise and unambiguous implementation of the Witten–Veneziano formula can be obtained [8].

These theoretical developments generated a renewed interest in 't Hooft's results and prompted us to perform an exploratory numerical investigation of a class of two dimensional non-Abelian models with overlap lattice fermions. Precisely, we have simulated models of QCD_2 with $U(N_c)$ and $SU(N_c)$ color groups for $N_c = 2, 3, 4$ ($N_f = 0, 1, 2$) imposing the 't Hooft's condition $C_t = g^2 N_c = \text{const.}$ Our systems are small enough that we could compute the fermionic propagator D^{-1} and $\det(D)$ exactly following the scheme used in [9] (see Refs. [9, 10] and [11, 12, 13, 14] for more refined implementations in two and four dimensions).

For $U(N_c)$ models we have found many background gauge configurations with zero modes in the fermionic operator. By counting them and averaging over the configurations we have computed the quenched topological susceptibility obtaining values in very good agreement with the analytic results. We have computed the chiral condensate for $N_f = 1$ which turns out to be compatible with a non-zero value. For $SU(N_c)$ models we have not found any configuration with exact zero modes as expected since these models have an exact $U_A(1)$ symmetry in the chiral limit.

From two-point correlation functions of fermion bilinears we have extracted the meson masses and the corresponding decay constants. In the $U(N_c)$ case the η' mass in the chiral limit verifies the Witten-Veneziano relation [15, 16] within errors for each N_c . The pion masses verify quite well the expected functional dependence $M_\pi^2 \propto m_q$ [2]. For $SU(N_c)$ models our data favor the functional dependence $M_\pi \propto m_q^{2/3}$ [17, 18]. In both cases, at fixed N_f , data exhibit universality in N_c and quenched results get closer and closer to unquenched ones when N_c increases.

In the next section we shall briefly remind the reader of some properties of the 't Hooft model in the continuum. In section 3 we define the overlap regularization we have imple-

mented numerically. In section 4 we present our numerical results for the meson spectra and decay constants. In Section 5 we compute the topological susceptibility and we compare the η' mass with the one extracted from the Witten-Veneziano formula. In section 6 we report our results for the chiral condensates. Section 7 is devoted to some concluding remarks.

2 The 't Hooft Model in the Continuum

We consider two-dimensional models with color group $U(N_c)$ and $SU(N_c)$ with N_f degenerate flavors defined by the action

$$S = \int d^2x \left[\frac{1}{2} \text{tr} F_{\mu\nu} F_{\mu\nu} + \sum_{i=1, N_f} \bar{\psi}_i (\gamma_\mu D_\mu + m_q) \psi_i \right] \quad (1)$$

The ψ is a N_f -dimensional fermion multiplet¹ and we use the following representation of the two-dimensional γ -matrices

$$\gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_2, \quad \gamma_5 = -i\gamma_1\gamma_2 = \sigma_3 \quad (2)$$

where σ_i are the Pauli matrices. In Eq. (1) $D_\mu = \partial_\mu + igA_\mu$, where $A_\mu = A_\mu^A t^A$ is the gauge potential and t^A are the N_c^2 or $N_c^2 - 1$ generators for the groups $U(N_c)$ or $SU(N_c)$ respectively, normalized according to $\text{tr}(t^A t^B) = 1/2 \delta^{AB}$. The field strength reads $F_{\mu\nu} = F_{\mu\nu}^A t^A$, where

$$F_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - gf^{ABC} A_\mu^B A_\nu^C \quad (3)$$

The models described in Eq. (1) are super-renormalizable and therefore g and m_q are finite bare parameters. 't Hooft studied the $U(N_c)$ models in the limit

$$N_c \rightarrow \infty, \quad g^2 N_c = C_t = \text{constant} \quad (4)$$

which corresponds to take only planar diagrams with no fermion loops [2].

2.1 The $U(N_c)$ models

The massless action in Eq. (1) has an $U_V(N_f) \otimes U_A(N_f)$ flavor symmetry. The $U_V(1)$ symmetry is preserved while the $U_A(1)$ is softly broken by the quark mass m_q and explicitly broken by the anomaly which in two dimensions appears in two-point functions, not in triangle loops [2]. The corresponding singlet Ward identities are

$$\partial_\mu V_\mu(x) = 0 \quad (5)$$

$$\partial_\mu A_\mu(x) = 2N_f Q(x) + 2m_q P(x) \quad (6)$$

¹Color and spinor indices are suppressed throughout the paper.

where

$$V_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x) , \quad A_\mu(x) = -i\epsilon_{\mu\nu}V_\nu(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x) , \quad P(x) = \bar{\psi}(x)\gamma_5\psi(x) \quad (7)$$

and the topological charge density reads

$$Q(x) = \frac{g\sqrt{N_c}}{4\pi} \epsilon_{\mu\nu}F_{\mu\nu}^0(x) \quad (8)$$

where $F_{\mu\nu}^0$ is the Abelian field strength. The Ward identities associated to the non-singlet ($N_f > 1$) axial and vector symmetries are given by

$$\partial_\mu V_\mu^f(x) = 0 \quad (9)$$

$$\partial_\mu A_\mu^f(x) = 2m_q P^f(x) \quad (10)$$

where V_μ^f , A_μ^f and P_μ^f are defined as in Eq.(7), but with the insertion of the proper flavor generator.

The topological charge of a $U(N_c)$ background gauge configuration is

$$Q = \int d^2x Q(x) = \frac{g\sqrt{N_c}}{4\pi} \int d^2x \epsilon_{\mu\nu}F_{\mu\nu}^0(x) \quad (11)$$

and it is related to the difference between the number of positive (n_+) and negative (n_-) eigenvalues of the Dirac operator through the Atiyah-Singer theorem

$$Q = n_- - n_+ \quad (12)$$

In two dimensions the vanishing theorem ensures that only n_+ or n_- is non-zero [19]. The topological susceptibility χ in the pure gauge theory is [20]

$$\chi = \int d^2x \langle Q(x)Q(0) \rangle|_{YM} = \frac{C_t}{4\pi^2} \quad (13)$$

where the expectation values $\langle \dots \rangle|_{YM}$ in Eq. (13) have been taken in the theory without fermion fields.

Since the $U_A(1)$ symmetry is anomalous, for $N_f = 1$ one can have a non-zero chiral condensate [2, 21] $\langle \bar{\psi}\psi \rangle$ without violating the Mermin-Wagner theorem [22]. It is related to the topological susceptibility in the full theory as

$$\chi = -m_q \langle \bar{\psi}\psi \rangle + O(m_q^2) \quad (14)$$

On the contrary, for $N_f > 1$ a non-zero condensate would spontaneously break the $SU_A(N_f)$ symmetry.

The meson spectrum of the two-flavor $U(N_c)$ models exhibits a pseudoscalar flavor-singlet excitation (η') and flavor-triplet quark-antiquark bound states (pions). The η' is massive due to the anomaly [2] and its mass in the chiral limit is

$$M_{\eta'}^2 = N_f \frac{g^2}{\pi} = N_f \frac{C_t}{N_c} \frac{1}{\pi} \quad (15)$$

The Witten-Veneziano formula [15, 16] for the $U(N_c)$ models gives

$$M_{\eta'}^2 = \frac{4N_f}{f_\pi^2} \chi \quad (16)$$

where f_π is the pion decay constant. By inverting Eq.(16), assuming that it is an exact relation between the η' mass and the topological susceptibility and taking into account Eqs. (13), (15) and (16) one gets

$$f_\pi = \sqrt{\frac{N_c}{\pi}} \quad (17)$$

In the $N_c \rightarrow \infty$ limit and for $m_q \ll \sqrt{C_t/\pi}$, the pion mass M_π as a function of the quark mass m_q reads [2]

$$M_\pi^2 = 2 \sqrt{\frac{C_t \pi}{3}} m_q + \dots \quad (18)$$

It is interesting to note that M_π^2 is linear at the leading order in the quark mass like in four-dimensional QCD and the coefficient in front of m_q is expected to be exact in the $N_c \rightarrow \infty$ limit [2].

2.2 The $SU(N_c)$ models

Analogously to the previous case, the massless action (1) of two dimensional $SU(N_c)$ models has an exact $U_V(N_f) \otimes U_A(N_f)$ flavor symmetry. Since in this case there is no Abelian field strength component $F_{\mu\nu}^0$, the $U_A(1)$ symmetry is only softly broken by the quark mass m_q . Therefore the Dirac operator should not have any zero-modes.

In the limit $N_c \rightarrow \infty$, a non-zero chiral condensate was obtained in [23]

$$\langle \bar{\psi}\psi \rangle = -N_c \left(\frac{g^2 N_c}{12\pi} \right)^{\frac{1}{2}} = -\frac{g}{\sqrt{12\pi}} N_c^{\frac{3}{2}} \quad (19)$$

and a Berezinski-Kosterlitz-Thouless phase transition was advocated to reconcile this result with the Mermin-Wagner theorem [22]. This behavior would favor the argument that in the limit $N_c \rightarrow \infty$ both $U(N_c)$ and $SU(N_c)$ gauge groups should describe the same physics [1, 24, 25].

The meson spectrum of the $SU(N_c)$ models exhibits only pions. For $N_f = 1$ and the $SU(2)$ color group their mass has been computed in [18] in the semiclassical WKB approximation

$$M_\pi^2 = \frac{9}{\pi} (2^7 C_t)^{\frac{1}{3}} \left(\frac{e^\gamma}{\pi}\right)^{\frac{4}{3}} m_q^{\frac{4}{3}} + \dots \quad (20)$$

3 The Overlap Dirac operator

We have implemented the lattice action

$$S = S_G(U) + \sum_{i=1, N_f} \sum_{x, y} \bar{\psi}_i(x) D_{m_q}(x, y) \psi_i(y) \quad (21)$$

where $S_G(U)$ is the standard Wilson gauge action

$$S_G(U) = \beta \sum_{x, \mu < \nu} \left[1 - \frac{1}{2N_c} \text{Tr} (U_{\mu\nu}(x) + U_{\mu\nu}^\dagger(x)) \right] \quad (22)$$

$\beta = 2N_c/(ga)^2$, a and g being the lattice spacing and bare coupling constant and

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x + a\hat{\mu}) U_\mu^\dagger(x + a\hat{\nu}) U_\nu^\dagger(x) \quad (23)$$

In Eq. (21)

$$D_{m_q}(x, y) = \left[\left(1 - \frac{m_q a}{2}\right) D(x, y) + m_q \delta_{xy} \right] \quad (24)$$

where the Neuberger-Dirac operator is defined as [3]

$$D \equiv \frac{1}{a} \left(1 + (D_W - \frac{1}{a}) \left[(D_W^\dagger - \frac{1}{a})(D_W - \frac{1}{a}) \right]^{-\frac{1}{2}} \right) \quad (25)$$

D_W is the Wilson-Dirac operator

$$D_W = \frac{1}{2} \gamma_\mu (\nabla_\mu + \nabla_\mu^*) - \frac{1}{2} a \nabla_\mu^* \nabla_\mu \quad (26)$$

where

$$\nabla_\mu \psi_i(x) = \frac{1}{a} [U_\mu(x) \psi_i(x + a\hat{\mu}) - \psi_i(x)] \quad (27)$$

$$\nabla_\mu^* \psi_i(x) = \frac{1}{a} [\psi_i(x) - U_\mu^\dagger(x - a\hat{\mu}) \psi_i(x - a\hat{\mu})] \quad (28)$$

The Neuberger Dirac operator is γ_5 -hermitian, i.e. $D^\dagger = \gamma_5 D \gamma_5$, and satisfies the Ginsparg-Wilson relation [26]

$$\gamma_5 D^{-1} + D^{-1} \gamma_5 = a \gamma_5 \quad (29)$$

which guarantees that in the chiral limit the lattice action (21) is invariant under the continuum symmetry [5]

$$\delta\psi_i = \gamma_5(1 - aD)\psi_i, \quad \delta\bar{\psi}_i = \bar{\psi}_i\gamma_5 \quad (30)$$

As a consequence the $U_A(1)$ anomaly, if present, is recovered à la Fujikawa [6] and the Dirac operator has exact chiral zero modes for topologically non-trivial gauge field configurations [7]. The analogous flavor non-singlet chiral transformations are obtained by including a flavor group generator in Eq. (30).

From Eq. (29) one can derive the following identities ($a = 1$)

$$D_{m_q}^\dagger D_{m_q} = \left(1 - \frac{m_q^2}{4}\right) [D + D^\dagger] + m_q^2 \quad (31)$$

$$D_{m_q}^{-1} = \left(\frac{1}{1 + \frac{m_q}{2}} D^\dagger + \frac{m_q}{1 - \frac{m_q^2}{4}} \right) \frac{1}{[D + D^\dagger] + m_q^2 \left(1 - \frac{m_q^2}{4}\right)^{-1}} \quad (32)$$

which turns out to be useful in the numerical implementation.

4 The Pion Masses and Decay Constants

In order to study the response of the overlap Dirac operator in the presence and absence of the chiral anomaly and at the same time to analyze the scaling of the physical observables with N_c (at fixed N_f), we have performed extensive simulations of $U(N_c)$ and $SU(N_c)$ models for $N_c = 2, 3, 4$. We have generated the gauge configurations with a standard Metropolis Monte Carlo algorithm according to the gauge action in Eq. (22). To avoid the Gross-Witten phase transition [27] we have chosen $C_t = g^2 N_c = 4/3$, which corresponds to $\beta = 6$ for $N_c = 2$, $\beta = 13.5$ for $N_c = 3$ and $\beta = 24$ for $N_c = 4$. We have generated 500 independent configurations for $N_c = 2$, 300 for $N_c = 3$ and 150 for $N_c = 4$ separated by 10000 sweeps of the whole lattice. For all gauge groups we have fixed the same dimensions in lattice units, i.e. $N_t = N_x = 18$. Consequently the Neuberger-Dirac operator is a complex matrix of dimension 1296×1296 for $N_c = 2$, 1944×1944 for $N_c = 3$ and 2592×2592 for $N_c = 4$. We could diagonalize exactly the hermitian operator $D + D^\dagger$ by using full matrix algebra routines with the resources available to us. By using Eq. (31) we have computed the eigenvalues of the massive Neuberger operator and its determinant and with Eq. (32) we have determined the propagators of the massive fermions. We have explicitly checked that, for each of the fermionic masses $m_q = 0.04, 0.05, 0.06, 0.07, 0.08, 0.1$, the lattice spans at least four pion correlation lengths. The effects of dynamical fermions have been included by weighting the observables with the appropriate powers of the fermion determinant. The smallness of the 18×18 lattice warrants this procedure [9], which would lead to an unacceptable large variance on larger systems.

The meson masses and the decay constants have been extracted in the standard manner from the vector correlator at zero momentum

$$\sum_{x,x',y} \langle V_1(x,y) V_1(x',y+t) \rangle \quad (33)$$

where the non-singlet vector current we have used is

$$V_\mu(x,y) = \bar{\psi}_1(x,y) \gamma_\mu \left[\left(1 - \frac{a}{2} D\right) \psi_2 \right](x,y) \quad (34)$$

In Table 1 we report the pion mass squared for the $U(N_c)$ models with two flavors of

$m_q/g\sqrt{N_c}$	$N_c = 2$ $M_\pi^2/g^2 N_c$	$N_c = 3$ $M_\pi^2/g^2 N_c$	$N_c = 4$ $M_\pi^2/g^2 N_c$
$N_f = 2$			
0.0346	0.046(10)	0.047(3)	0.055(6)
0.0433	0.056(10)	0.059(3)	0.067(5)
0.0519	0.068(8)	0.072(3)	0.080(4)
0.0606	0.080(7)	0.085(3)	0.093(3)
0.0692	0.094(6)	0.098(3)	0.107(3)
0.0866	0.124(4)	0.128(3)	0.138(2)
quenched ($N_f = 0$)			
0.0346	0.069(1)	0.063(1)	0.057(1)
0.0433	0.083(1)	0.077(1)	0.071(1)
0.0519	0.098(1)	0.091(1)	0.086(1)
0.0606	0.113(1)	0.106(1)	0.100(1)
0.0692	0.129(1)	0.121(1)	0.116(1)
0.0866	0.161(1)	0.153(1)	0.148(1)

Table 1: $M_\pi^2/g^2 N_c$ vs. $m_q/g\sqrt{N_c}$, $U(N_c)$ models.

dynamical fermions and in the quenched approximation. The quenched results get closer to the unquenched two-flavor ones when N_c gets larger. Figure 1 shows M_π^2 as a function of the quark mass for $N_f = 2$ and provides evidence of universality in N_c for the pion masses. According to Eq. (18), M_π^2 should have a linear dependence in m_q and it is expected to vanish in the chiral limit. From a fit

$$\frac{M_\pi^2}{g^2 N_c} = A + B \frac{m_q}{g\sqrt{N_c}} \quad (35)$$

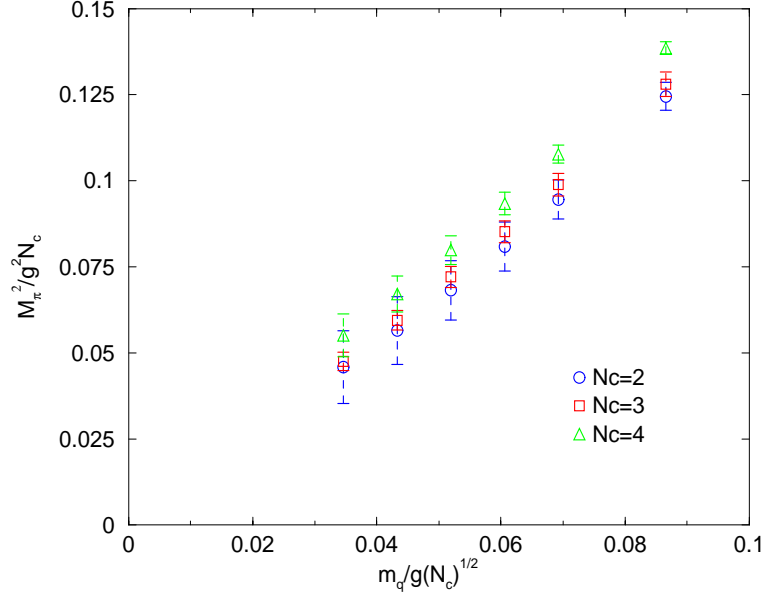


Figure 1: $M_\pi^2/g^2 N_c$ vs. $m_q/g\sqrt{N_c}$ for $N_f = 2$, $U(N_c)$ models.

we obtained $A = -0.013(17)$ and $B = 1.57(19)$ for $N_c = 2$, $A = -0.007(4)$ and $B = 1.54(4)$ for $N_c = 3$ and $A = -0.006(8)$ and $B = 1.66(9)$ for $N_c = 4$. In order to carefully determine the systematic error affecting B , further simulations would be necessary that go beyond the scope of the present investigation.

The pion decay constants f_π turns out to be almost constant in m_q within our statistical errors. Their values in the chiral limit are reported in Table 2.

$N_c = 2$	$N_c = 3$	$N_c = 4$
f_π	f_π	f_π
$N_f = 2$		
0.7(1)	0.94(5)	1.10(1)
quenched ($N_f = 0$)		
0.77(1)	0.96(1)	1.12(1)

Table 2: f_π , $U(N_c)$ models.

In Table 3 we report the pion masses for the $SU(N_c)$ models with one and two flavor of

dynamical fermions and in the quenched approximation. Figure 2 shows the pion masses for

$(m_q/g\sqrt{N_c})^{2/3}$	$N_c = 2$ $M_\pi/g\sqrt{N_c}$	$N_c = 3$ $M_\pi/g\sqrt{N_c}$	$N_c = 4$ $M_\pi/g\sqrt{N_c}$
$N_f = 2$			
0.1063	0.183(15)	0.207(23)	0.181(28)
0.1233	0.211(12)	0.237(23)	0.212(26)
0.1392	0.236(10)	0.263(22)	0.240(24)
0.1543	0.259(8)	0.286(19)	0.267(23)
0.1687	0.280(7)	0.308(16)	0.292(21)
0.1957	0.321(5)	0.348(10)	0.341(18)
$N_f = 1$			
0.1063	0.182(7)	0.196(4)	0.199(12)
0.1233	0.211(6)	0.226(4)	0.229(10)
0.1392	0.237(5)	0.254(4)	0.258(9)
0.1543	0.262(4)	0.279(3)	0.285(8)
0.1687	0.285(4)	0.303(3)	0.310(7)
0.1957	0.329(3)	0.349(2)	0.357(6)
quenched ($N_f = 0$)			
0.1063	0.201(3)	0.212(1)	0.216(1)
0.1233	0.227(2)	0.241(1)	0.246(1)
0.1392	0.253(2)	0.268(1)	0.273(1)
0.1543	0.277(2)	0.293(1)	0.299(1)
0.1687	0.300(2)	0.316(1)	0.323(1)
0.1957	0.344(2)	0.361(1)	0.368(1)

Table 3: $M_\pi/g\sqrt{N_c}$ vs. $(m_q/g\sqrt{N_c})^{2/3}$, $SU(N_c)$ models.

$N_f = 2$ and reveals also in this case a universality in N_c already for $N_c = 2, 3, 4$. According to Eq. (20), M_π^2 should have an $m_q^{2/3}$ quark mass dependence and should vanish for $m_q = 0$. From a fit

$$\frac{M_\pi}{g\sqrt{N_c}} = A + B \left(\frac{m_q}{g\sqrt{N_c}} \right)^{\frac{2}{3}} \quad (36)$$

we got $A = 0.023(26)$ and $B = 1.52(12)$ for $N_c = 2$, $A = 0.046(43)$ and $B = 1.55(17)$ for $N_c = 3$ and $A = -0.01(4)$ and $B = 1.79(12)$ for $N_c = 4$, where again the errors are statistical only. We performed also a fit of the form

$$M_\pi/g\sqrt{N_c} = C \left(m/g\sqrt{N_c} \right)^\gamma \quad (37)$$

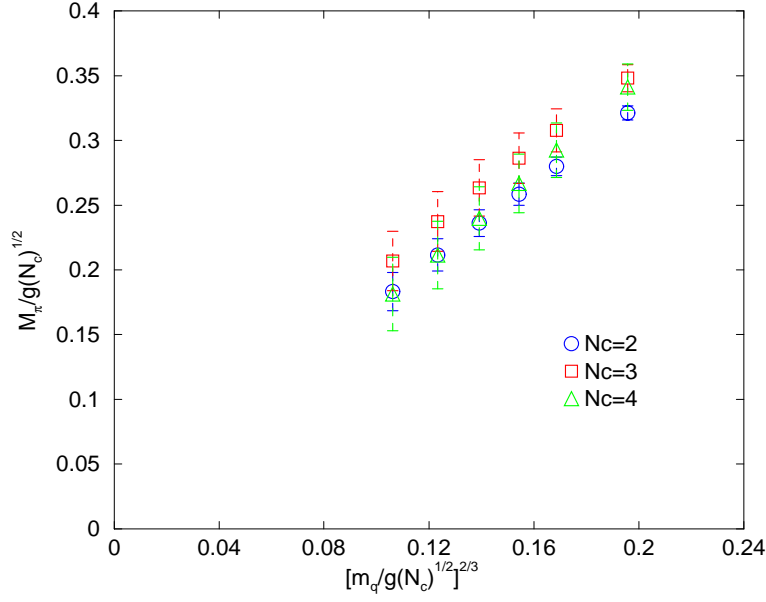


Figure 2: $M_\pi/g\sqrt{N_c}$ vs. $(m_q/g\sqrt{N_c})^{2/3}$ for $N_f = 2$, $SU(N_c)$ models.

and we obtained a value of γ compatible with $2/3$ for $N_c = 2, 3, 4$.

The values of the pion decay constants f_π in the chiral limit for the $SU(N_c)$ models are reported in Table 4.

$N_c = 2$	$N_c = 3$	$N_c = 4$
f_π	f_π	f_π
$N_f = 2$		
0.64(6)	0.92(3)	1.21(12)
$N_f = 1$		
0.70(2)	0.98(2)	1.14(3)
quenched ($N_f = 0$)		
0.76(1)	0.98(1)	1.12(1)

Table 4: f_π , $SU(N_c)$ models.

5 The Witten-Veneziano Relation and the η' Mass

The topological charge of a given background configuration can be computed by counting the number of zero modes of the overlap Dirac operator. Since we have diagonalized exactly the overlap operator, we could compute the quenched topological susceptibility χ by averaging the square of the topological charge, normalized with the volume, over the configurations. In Table 5 we report the results we have obtained for the various N_c .

$N_c = 2$	$N_c = 3$	$N_c = 4$
χ	χ	χ
0.0258(16)	0.0298(23)	0.0319(37)

Table 5: Topological susceptibility, $U(N_c)$ models.

Even if with increasing N_c the topological susceptibility get closer and closer to the analytical value $\chi = 0.0337$ given in Eq. (13), more accurate studies of discretization and finite size effects, that goes beyond the scope of our exploratory investigation, are required to reach a final conclusion. In fact if the magnitude of these systematic uncertainties is different for the gauge groups, they would spoil any extrapolation to $N_c \rightarrow \infty$. For example, building on a trade-off between spatial and internal degrees of freedom that prompted the introduction of the $N_c \rightarrow \infty$ single plaquette model [29], one could heuristically argue that the theory should be less affected by finite size effects for increasing N_c . To try to estimate finite size effects in the worst case, we computed the value of the topological susceptibility of the $U(2)$ model for the volume $V = 24 \times 24$ and we obtained $\chi = 0.0316(24)$. The difference of this value with the analogous one in Table 5 gives a rough estimate of the systematic error induced by finite volume effects on χ .

For $SU(N_c)$ model we directly checked that there are no zero modes for each gauge field configuration and therefore the topological charge is always zero.

From the values of χ reported in Table 5 and the pion decay constants at $N_f = 0$ given in Table 2, we have computed $M_{\eta'}^2$ for $N_f = 1$ given in Table 6 (statistical errors only). It turns out that the WV relation is satisfied for all $U(N_c)$ gauge groups within our overall uncertainties.

The singlet pseudoscalar correlation functions are given by differences between connected and disconnected contributions and therefore they are noisier with respect to the non-singlet case. To have an independent determination of the η' mass for $N_f = 1$, we resorted the

$N_c = 2$ $M_{\eta'}^2/g^2 N_c$	$N_c = 3$ $M_{\eta'}^2/g^2 N_c$	$N_c = 4$ $M_{\eta'}^2/g^2 N_c$
Analytic		
0.159	0.106	0.079
Numerical from Eq.(16)		
0.129(6)	0.097(7)	0.076(6)
Numerical from Eq.(40)		
0.159(8)	0.111(7)	0.084(8)

Table 6: $M_{\eta'}^2$ from the Witten-Veneziano relation in Eq. (16) and from Eq.(40).

method proposed in [28]. It exploits the quenched two-loop disconnected $\Gamma_q^{2\text{-loop}}$ and one-loop connected $\Gamma_q^{1\text{-loop}}$ contributions to the η' propagator

$$\Gamma_q(t) = \int dx \langle P(x, t) P(0, 0) \rangle \Big|_{\text{quenched}} \quad (38)$$

where the singlet pseudoscalar density is defined as

$$P(x, t) = \bar{\psi}(x, t) \gamma_5 \left[\left(1 - \frac{a}{2} D\right) \psi \right](x, t) \quad (39)$$

The formula we have used reads [28]

$$M_{\eta'}^2 = 2M_\pi \lim_{t \rightarrow \infty} \frac{\Gamma_q^{2\text{-loop}}(t)}{|t| \Gamma_q^{1\text{-loop}}(t)} \quad (40)$$

The results we have obtained, reported in Tab. 6, compares remarkably well with the analytic result in the chiral limit.

6 Chiral condensate

The chiral condensate can be expressed in terms of the eigenvalues $\lambda_i = 1 + e^{i\theta_i}$ of the Neuberger-Dirac operator as ($a = 1$)

$$\Sigma = -\frac{1}{N_f g \sqrt{N_c}} \langle \bar{\psi} \left(1 - \frac{1}{2} D\right) \psi \rangle = \frac{1}{g \sqrt{N_c} V} \frac{\langle (\det D_{m_q})^{N_f} F(Q, m_q, \cos \theta_i) \rangle_U}{\langle (\det D_{m_q})^{N_f} \rangle_U} \quad (41)$$

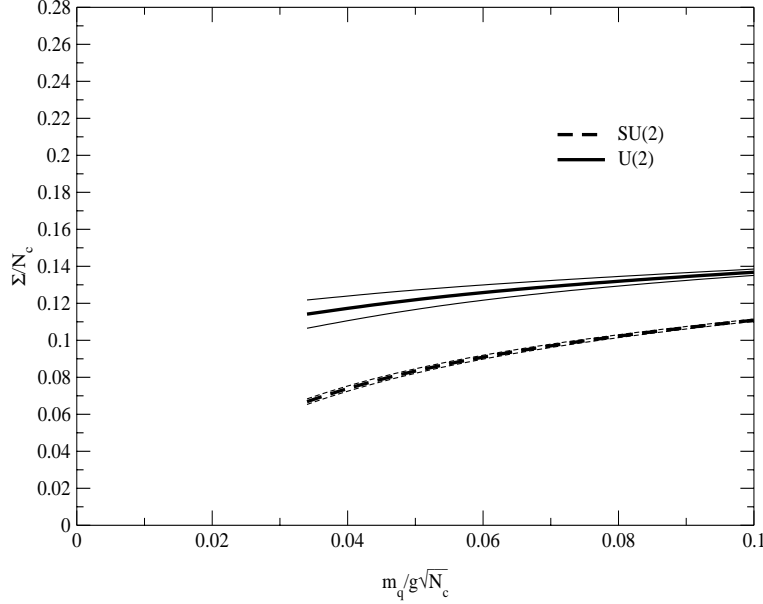


Figure 3: $N_f = 1$ chiral condensate for $U(2)$ and $SU(2)$.

where

$$F(Q, m_q, \cos \theta_i) = \frac{|Q|}{m_q} + \frac{m_q}{2} \sum_{i, \cos \theta_i \neq -1} \frac{1 - \cos \theta_i}{\left(1 + \frac{m_q^2}{4}\right) + \left(1 - \frac{m_q^2}{4}\right) \cos \theta_i} \quad (42)$$

and using Eq. (31)

$$(\det D_{m_q})^{N_f} = m_q^{N_f |Q|} \prod_{i, \cos \theta_i \neq -1} \left[\left(1 - \frac{m_q^2}{4}\right) 2(1 + \cos \theta_i) + m_q^2 \right]^{\frac{N_f}{2}} \quad (43)$$

For all the groups we have computed the chiral condensate by using the direct definition in Eq. (41). As a representative example, in Fig. 3 we compare the values of the $U(2)$ and $SU(2)$ chiral condensates for $N_f = 1$ computed up to $m_q/g\sqrt{N_c} = 0.0346$ in order to limit the finite size effects. A chiral extrapolation of these plots hints to a finite and zero value of the $U(2)$ and $SU(2)$ chiral condensates, respectively.

In Fig. 4 we show the chiral condensate for all $U(N_c)$ models with $N_f = 1$. They have been computed by using the definition in Eq. (41) and the one from the Axial Ward Identity in Eq. (14) obtained by neglecting the $O(m_q^2)$. The comparison is interesting because the mass dependence of the two quantities is different. Again both the determinations point in the direction of non-zero condensates. An interesting improvement in the determination of the condensates in the chiral limit could be obtained by implementing in two dimensions a

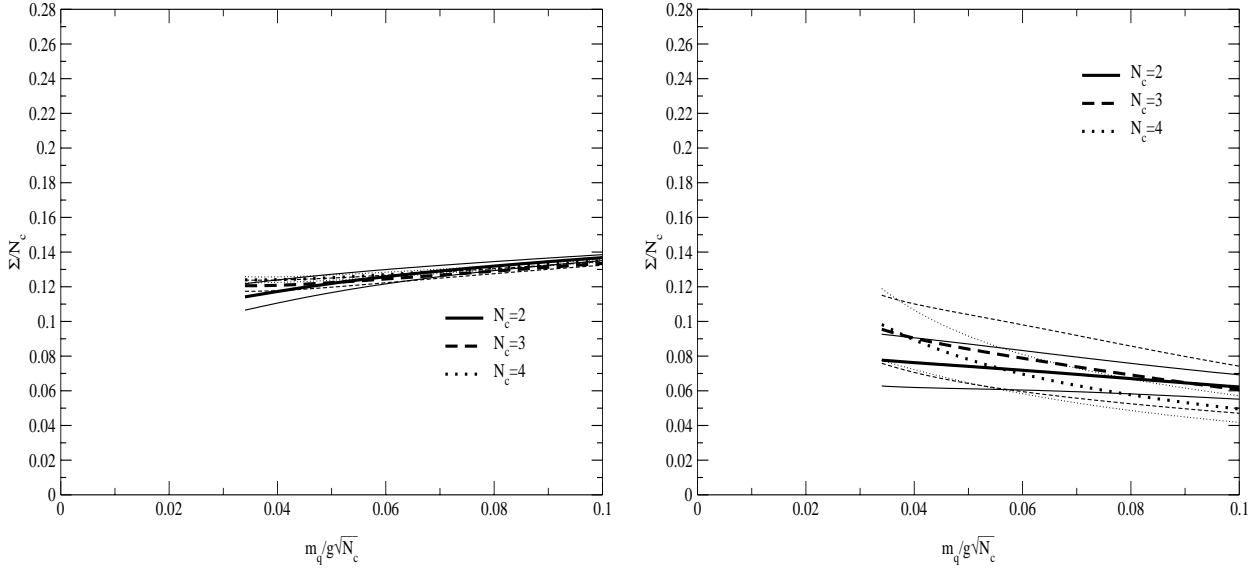


Figure 4: $N_f = 1$ chiral condensate. Left: from Eq. (41). Right: from Eq. (14).

finite volume technique analogous to the one proposed in four dimensions in Refs. [30, 31]. Moreover the use of algorithms which generate directly dynamical configurations could help in reducing the fluctuations induced by the re-weighting of the observables with the fermionic determinant.

7 Conclusions

We have performed an exploratory numerical study on the lattice of two dimensional models defined by the gauge groups $U(N_c)$ and $SU(N_c)$ ($N_c = 2, 3, 4$) and $N_f = 0, 1, 2$ degenerate fermions introduced by using the Neuberger-Dirac operator. Our results prove that the computation is feasible and it would be interesting to further pursue this line of research with a more detailed analysis, especially of discretization and finite size effects, that goes beyond the scope of our present investigation.

We have found that within our statistical errors the pion masses verify quite well the expected functional dependence $M_\pi^2 \propto m_q$ and $M_\pi \propto m_q^{2/3}$ for $U(N_c)$ and $SU(N_c)$ models respectively. In both cases, at fixed N_f , data exhibit universality in N_c and quenched results get closer and closer to unquenched ones when N_c increases.

As expected from an analysis of the symmetries of the models, for $SU(N_c)$ groups we

have not found any background gauge configuration with exact zero modes in the fermionic operator. On the other hand, many background gauge configurations with exact zero modes were found for the $U(N_c)$ models. By counting them and averaging over the configurations we have computed the quenched topological susceptibility obtaining values in good agreement with the analytic results. By using the meson decay constants extracted from the two-point functions, our data verify the Witten-Veneziano relation within errors for each N_c . Even if we could not safely extrapolate our data to the chiral limit, we have produced evidences that the chiral condensate for $N_f = 1$ is compatible with a non-zero value.

Of course, one would be interested to know how close real QCD is to the large N_c limit. In Refs. [32] it was pointed out, on the basis of a study of three and four dimensional pure gauge theories, that even the $SU(2)$ color group is close to $SU(\infty)$. Although our analysis is limited to QCD_2 , remarkably enough we saw that even with dynamical quarks 2, 3 and 4 colors appear to be in the same universality class, i.e. the physical quantities are degenerate in N_c within the errors.

Although our study is exploratory, the results we have obtained are very gratifying and indicate that it would be interesting to perform a deeper analysis in two dimensions and eventually extend it to four dimensions. In particular one could study within our computational scheme the baryons that are expected to be the QCD solitons [24]. An interesting improvement in the determination of the condensates in the chiral limit could be obtained by implementing finite volume techniques in two dimensions. Moreover the use of algorithms which generate directly dynamical configurations could help in reducing the fluctuations induced by the re-weighting of the observables with the fermionic determinant and therefore would allow to simulate larger lattices.

Acknowledgments

We warmly thank G.C. Rossi for many illuminating discussions. We wish to thank A. Gonzalez-Arroyo and H. Neuberger for interesting discussions. This work has been supported in part under DOE grant DE-FG02-91ER40676. The work of F.B. is supported in part by an INFN Postdoctoral Fellowship. He thanks the members of the Theoretical Particle Physics Group at Boston University for their kind hospitality.

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